ELECTRONIC SUPPLEMENTARY FILE

APPENDICES FOR: STATIONARY ROTARY FORCE WAVES ON THE LIQUID-AIR CORE INTERFACE OF A SWIRL ATOMIZER

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A. The Phase Velocity of Long Shallow Water Gravity Waves

This analysis is conducted in the cartesian coordinate system which has components x, y, and z with the fluid velocity given by V = u, v, w. The coordinate system is positioned such that the x and y axes are in the horizontal plane and the positive z axis is pointing vertically upwards, with z = 0 at the mean free surface of the liquid. The wave height η is considered to be of an asymptotic power series in some small parameter ε, which is representative of the maximum wave slope, Stoker [5]:

$$\eta(x, t) = \varepsilon\eta_0 + \varepsilon^2\eta_1 + \varepsilon^3\eta_2 + \ldots. \quad \text{(A1)}$$

The velocity is also written as an asymptotic power series in the small parameter ε:

$$\mathbf{V} = \varepsilon\mathbf{V}_0 + \varepsilon^2\mathbf{V}_1 + \varepsilon^3\mathbf{V}_2 + \ldots. \quad \text{(A2)}$$

In this way the analysis can be thought of as dealing with the first term of the series only. The use of the asymptotic power series then makes nonlinear terms of second order in ε, i.e. of order ε^2, so that they are considered to be negligible in comparison to linear terms.

The boundary conditions at the top and bottom surfaces are given by S(x,y,z,t)=0. At the free surface, or water air interface, the wave height will be at z = η(x, y, t) so that the boundary condition there is S = η – z =0. At the fixed surface or seabed, for ocean waves, z = -h so that the boundary condition there is S = z + h = 0. The analysis by Crapper [3], from which this section is adapted, begins with a general form of h = h(x,y), ie the depth of the seabed will differ from place to place. Later Crapper makes the simplifying assumption, in developing the wave equation as reproduced here, that the bottom is uniform and that h is in fact an absolute constant. The analysis simplifies considerably if one makes this simplifying assumption from the outset, and so this is done here. In fact what will be achieved here is essentially a one dimensional analysis and y-terms could actually be removed at this stage.

In developing the wave equation, for long waves in shallow water, the following three equations are employed, the total (particle or convective) time derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(\underline{V}.\,\nabla\right) = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \quad \text{(A3)}$$

the gradient of the time dependent version of the Bernoulli equation

$$\nabla\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}V^2 + \frac{P}{\rho_L} + gz\right) = \frac{\partial V}{\partial t} + \nabla\left(\frac{1}{2}V^2 + \frac{P}{\rho_L} + gz\right) = 0 \quad \text{(A4)}$$

(where ϕ is the velocity potential) and the equation of continuity, for an incompressible liquid,

$$\nabla.\underline{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad \text{(A5)}$$

As the top and bottom surface boundary conditions are S = 0 then the total derivative of the parameter S, using (A3), gives DS/Dt = 0 thus on S = η - z = 0, or z =η,

$$\frac{DS}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} - w = 0. \qquad \text{(A6)}$$

Equation (A6) contains two nonlinear terms which are second order in the small parameter ε and so will be negligible in comparison to the remaining terms. As the value of w (A6), occurs on the surface, z = η, then it will be denoted as such by the use of a subscript. Hence

$$\frac{\partial \eta}{\partial t} = w_{z=\eta}. \qquad \text{(A7)}$$

The continuity equation, (A5), may be rewritten as an integral to provide

$$w_{(z=\eta)} - w_{(z=-h)} = \int_{z=-h}^{z=\eta}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dx = 0. \qquad \text{(A8)}$$

As u and v are independent of z and $w_{(z=-h)} = 0$, there is no motion normal to the seabed at the seabed, then (A8) produces

$$w_{(z=\eta)} = -(\eta + h)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \qquad \text{(A9)}$$

As a coefficient η is small in comparison to h, in addition it would produce second order terms in the small parameter ε when multiplied by the velocity gradients, the η + h may therefore be reduced to h. The resultant expression, on the RHS of (A9) may then replace the RHS of (A7) to give

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \qquad \text{(A10)}$$

This will be left for the moment and attention will be turned to the pressure.

For the water, in a quiescent state, the pressure at a point z below the surface will be given by $p = \rho_L gz$, where ρ_L is the liquid density. If however the surface of the water has a wave of height η then the pressure at a point z + η below the free surface will have a pressure given by

$$p = \rho_L g(z + \eta). \qquad \text{(A11)}$$

The Bernoulli equation, (A4), can be applied to the wave problem with the pressure given as in (A11). As (A11) already contains the gravity term g then the elevation head term in (A4) is taken up by this parameter. Hence the u and v terms of (A4) become

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial \eta}{\partial x}=0 \qquad \text{(A12)}$$

and

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+g\frac{\partial \eta}{\partial y}=0. \qquad \text{(A13)}$$

The nonlinear middle terms of both of (A12) and (A13) are second order in the small parameter ε and hence negligible so these equations may be rewritten as

$$\frac{\partial u}{\partial t}+g\frac{\partial \eta}{\partial x}=0 \qquad \text{(A14)}$$

and

$$\frac{\partial v}{\partial t}+g\frac{\partial \eta}{\partial y}=0. \qquad \text{(A15)}$$

Next (A10) may be differentiated w.r.t. t, (A14) differentiated w.r.t. x and (A15)) differentiated w.r.t. y. The results of these actions may be combined to form

$$\frac{\partial^2 \eta}{\partial t^2}-gh\left(\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial^2 \eta}{\partial y^2}\right)=0. \qquad \text{(A16)}$$

This is a 'Wave Equation', it describes how the wave amplitude η behaves both in terms of time t and position in the x, y plane on the water surface. The phase velocity of the wave χ will be given by

$$\chi=\sqrt{gh}. \qquad \text{(A17)}$$

B. Weir Flow

The volume flow over the crest of a weir is the same as it is in the deeper upstream region of the river. However, the flow is faster due to the reduced cross-sectional area of the water. The remarkable phenomenon is that the surface level of the water is actually lower, over the top of the weir than it is in the main body of the water in the upstream region. This height, and the flow velocity, adjust optimally so that the rate is a maximum. Similarly, for a swirl atomizer, the air core in the outlet is bigger than in the swirl chamber. In this way the swirl atomizer acts as an axisymmetric weir with centrifugal force instead of gravity. The flow over a weir is depicted in figure 1 going from left to right within a channel. The flow takes place in a channel with a level bottom. The liquid in the

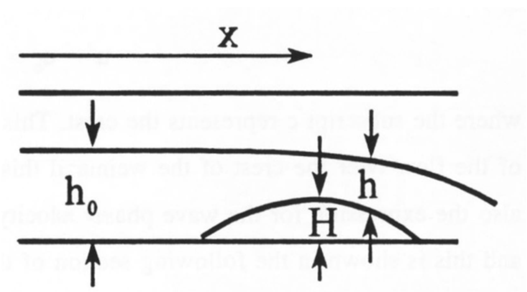

Figure 1. Flow from left to right over a weir.

main channel is maintained at a constant height h above the bottom. The weir is in fact a hump placed on the bottom of height H. The depth of the liquid, from the free surface down to the weir is h(x). The constant velocity for the liquid in the main channel is denoted u_0 and that flowing over the weir u(x). The methodology works by first determining an expression for the streaming velocity of the liquid passing over the crest of the weir u_c and then, secondly, by obtaining an expression for $\partial Q/\partial h$ into which is substituted this expression for u_c This then gives that $\partial Q/\partial h = 0$ at the crest of the weir showing, through the usual method of determining a maximum or minimum, that the depth of liquid h(x), at the crest, will be such as to permit maximum flow.

First Part

The first part of this analysis, to determine the velocity over the crest of the weir, begins with the Bernoulli equation which for the free surface streamline of this flow is

$$h(x) + H(x) + \frac{u^2(x)}{2g} = h_0 + \frac{u_o^2}{2g}. \tag{B1}$$

The volumetric flow rate Q, over the weir, is given by

$$Q = uhB. \tag{B2}$$

where B is the breadth of the channel. Here it is assumed that the channel has parallel sides so that B does not vary either in the vertical direction or the direction of streaming, x. The differentiation of (B1) and (B2) w.r.t. x give

$$\frac{dh}{dx} + \frac{dH}{dx} + \frac{u}{g}\frac{\partial u}{\partial x} = 0 \tag{B3}$$

and

$$\frac{\partial Q}{\partial x} = Bu\frac{dh}{dx} + Bh\frac{\partial u}{\partial x} = 0 \tag{B4}$$

which is equal to zero as Q will not vary with x, the direction of streaming. Both (B3) and (B4) may be rearranged to make $\partial h/\partial x$ the subject. The resultant expressions are equated with one another to provide

$$\left(\frac{u}{g} - \frac{h}{u}\right)\frac{\partial u}{\partial x} + \frac{dH}{dx} = 0. \tag{B5}$$

At the crest of the weir, dH/dx = 0. The streaming velocity u(x) will continue to increase with axial direction as it passes over the weir so that $\partial h/\partial x \neq 0$, at the axial position of the crest of the weir. If all of these, reasonable, assumptions are applied to (B5) then it only remains that, at the crest

$$u = u_c = \sqrt{gh}. \tag{B6}$$

where the subscript c represents the crest. This establishes an expression for the velocity of the flow over the crest of the weir and this will be used presently. Equation (B6) is also the expression for the wave phase velocity of a long gravity wave in shallow water and this is shown in the previous appendix.

Second Part

The second part of this weir flow analysis is designed to establish the principle of maximum flow for a weir i.e. that the volumetric flow rate Q is a maximum as a function of the height h of the weir by showing that ∂Q/∂h = 0 at the weir crest. By rearranging (B1) to make u^2 the subject and employing (B2) one may derive

$$Q = \sqrt{2gh^2B^2\left(h_0 + \frac{u_0^2}{2g} - h - H\right)}. \qquad \text{(B7)}$$

So that, as B, h_0 and u_0 are constants and H is not a function of h,

$$\frac{\partial Q}{\partial h} = \frac{1}{2}\left(2gB^2h^2\left[h_0 + \frac{u_0^2}{2g} - h - H\right]\right)^{-\frac{1}{2}} 2gB^2\left(2gB^2h^2\left[h_0 + \frac{u_0^2}{2g} - h - H\right] - h^2\right). \qquad \text{(B8)}$$

From (B1)

$$h_0 + \frac{u_0^2}{2g} - h - H = \frac{u_0^2}{2g} \qquad \text{(B9)}$$

so that, after some work, (B8) simplifies down to

$$\frac{\partial Q}{\partial h} = \frac{B}{u}\left(u^2 - gh\right). \qquad \text{(B10)}$$

Thus if $u = u_c = \sqrt{(gh)}$ at the crest of the weir, as derived in (B6), then ∂Q/∂h = 0. This establishes that the volumetric flow rate Q is a maximum as a function of h at the crest of the weir or, as by continuity Q is clearly a constant, that the height of the flow over the hump h(x), and the velocity u(x), adjusts themselves optimally.

C. Vorticity

This is fairly basic stuff but might save folks time if I present it here. To get the result wr = c, the tangential velocity times the radius at which that velocity takes place is a constant, we look at the curl of the velocity in cylindricals, x, r, θ and u, v, w :

$$\nabla \times \underline{V} = \left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{\partial w}{\partial x}\right)\underline{r} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial r}\right)\underline{\theta} + \frac{1}{r}\left(\frac{\partial (rw)}{\partial r} - \frac{\partial v}{\partial \theta}\right)\underline{x} = 0. \qquad \text{(C1)}$$

For the r – θ plane we look at the x –component. For irrotational flow the particles adjacent to one and other are independent or "free" and the "circulation" is zero (see for example http://en.wikipedia.org/wiki/Vortex) for an animation. Thus

$$\frac{\partial (rw)}{\partial r} - \frac{\partial v}{\partial \theta} = 0. \qquad \text{(C2)}$$

For axially symmetric flow, there is no variation w.r.t. θ, then

$$\frac{\partial(\mathrm{rw})}{\partial \mathrm{r}} = 0 \qquad \text{(C2)}$$

which leads directly to wr = const.