The Analogy between Waves on the Surface of the Air-core of a Swirl Atomizer and Long, Shallow Water, Gravity Waves.

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In recent years the phenomenon of waves occurring on the surface of the air-core within swirl atomizers has been reported from the result of experimental work. Many advanced textbooks on water waves provide a mathematical derivation of a wave equation for shallow water gravity waves. This paper demonstrates that this type of mathematical analysis may be adapted, by substituting a swirling centrifugal acceleration for gravity, in order to formulate a wave equation applicable to the waves occurring on the surface of the air-core within swirl atomizers.

1. Introduction

The phenomena of waves on liquid sheets issuing from atomizers, under relatively slow operating conditions, has been known for some time [1]. In particular, waves are the precursor to ligament and drop formation on the conical liquid sheets of swirl atomizers. More recently, it has been postulated that the sheet waves originate from further upstream, on the surface of the air-core within the body of the atomizer [2, 3].

A number of text books have been written on water-wave theory [4, 5, 1]. Within the pages of these volumes is an elementary analytical mathematical development of the wave equation for long shallow water gravity waves. Within the present paper is an analogical development of the wave equation for, what might be termed, centrifugal waves occurring on the air-core of a swirl atomizer.

The analysis, in the cylindrical coordinate system, is based on the Bernoulli equation for inviscid flows. An irrotational, in the mathematical sense, free-vortex flow is assumed. An asymptotic power series form is assumed for the wave height which is a function of the axial displacement and of time, $\eta(x, t)$.

2. **Long Centrifugal Waves on the Air-Core of a Swirling Flow Through a Nozzle**

This analysis was conducted in the cylindrical coordinate system (x, r, θ) with velocity components (u, v, w). The flow of fluid was deemed to be axially symmetric so that terms differentiated w.r.t. θ are zero. The radial velocity v, of the bulk flow, has been deemed to be zero in both the swirl chamber and the outlet orifice as the orifice wall provides an effective barrier to such movement. On the surface of the air-core there must clearly exist a local velocity normal to the mean free surface, in order to provide a wave amplitude, and so in this treatment the radial velocity has been retained. As with the treatment of gravity waves in the appendix, both the velocity **V** and the wave amplitude were considered to be of the form of an asymptotic power series in the small parameter ε, which is representative of the maximum wave slope, [5]:

$$\mathbf{V} = \varepsilon \mathbf{V}_0 + \varepsilon^2 \mathbf{V}_1 + \varepsilon^3 \mathbf{V}_2 + \dots \tag{1}$$

and

$$\eta(x, t) = \varepsilon \eta_0 + \varepsilon^2 \eta_1 + \varepsilon^3 \eta_2 + \dots \tag{2}$$

The fixed surface (wall) boundary condition is $r = r_w$, or $S(r, x) = r_w - r = 0$, and the free surface boundary condition is $r = r_{ac} - \eta$, or $S(r, x, t) = (r_{ac} - \eta) - r = 0$. As the top surface boundary condition is S = 0 then the total derivative of S will also be zero, there is no temporal or spatial variation of this condition, hence

$$\frac{DS}{Dt}=\frac{\partial S}{\partial t}+v\frac{\partial S}{\partial r}+u\frac{\partial S}{\partial x}= \\ \frac{\partial}{\partial t}[(r_{ac}-\eta)-r]+v\frac{\partial}{\partial r}[(r_{ac}-\eta)-r]+u\frac{\partial}{\partial x}[(r_{ac}-\eta)-r]=0, \quad (3)$$

on the top surface. On the understanding of which parameters are not functions of a particular variable then this may be simplified and rearranged to form

$$\frac{\partial\eta}{\partial t}-u\frac{\partial}{\partial x}(r_{ac}-\eta)=-v. \quad (4)$$

The term $\partial r_{ac}/\partial x$ is small compared to $\partial\eta/\partial t$ and the term $u\partial\eta/\partial x$ is second order in the small parameter ε, and hence also small. On neglecting these terms eqn.(4) may be reduced to

$$\frac{\partial\eta}{\partial t}=-v_{(r=r_{ac}-\eta)} \quad (5)$$

where the subscript has been added to v in order to denote that this is the condition on the free surface.

The continuity equation may be written in cylindrical coordinates, with axial symmetry, as

$$\nabla\,.\,\underline{V}=\frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{\partial u}{\partial x}=0. \quad (6)$$

This may be rearranged to form

$$\partial(rv)=-\frac{\partial u}{\partial x}r\,\partial r. \quad (7)$$

As $\partial u/\partial x$ is not a function of r then eqn.(7) may be integrated between the limits of the top and bottom surfaces, $r = r_{ac}-\eta$ to $r = r_w$, to give

$$r_w\,v_{(r=r_w)}-(r_{ac}-\eta)\,v_{(r=r_{ac}-\eta)}=-\frac{1}{2}\frac{\partial u}{\partial x}\left(r_w^2-(r_{ac}-\eta)^2\right). \quad (8)$$

With v = 0 at the wall and with both the remaining LHS and the RHS, of eqn.(8), divided through by $(r_{ac}-\eta)$ this gives

$$-v_{(r=r_{ac}-\eta)}=-\frac{1}{2}\frac{\partial u}{\partial x}\frac{\left(r_w^2-(r_{ac}-\eta)^2\right)}{(r_{ac}-\eta)}. \quad (9)$$

The RHS of eqn.(9) may be used to replace the RHS of eqn.(5) to give

$$\frac{\partial\eta}{\partial t}+\frac{1}{2}\left[\frac{r_w^2-(r_{ac}-\eta)^2}{r_{ac}-\eta}\right]\frac{\partial u}{\partial x}=0. \quad (10)$$

This will be left for the moment and attention will be turned to the pressure.

For a free vortex $p/\rho_L + w^2/2$ = constant, where p is the pressure at any point in the liquid and w is the tangential velocity of the spinning liquid, given by w = c/r, where c is the free-vortex constant. At the surface of the air-core, $r = r_{ac} - \eta$, the pressure is atmospheric and is taken as zero gauge pressure, p = 0, so that

$$0+\frac{c^2}{2(r_{ac}-\eta)^2}=\text{const}. \quad (11)$$

thus defining the constant. This gives

$$\frac{p}{\rho_L}+\frac{w^2}{2}=\frac{c^2}{2(r_{ac}-\eta)^2} \quad \text{or} \quad \frac{p}{\rho_L}=\frac{c^2}{2}\left(\frac{1}{(r_{ac}-\eta)^2}-\frac{1}{r^2}\right). \tag{12}$$

The time dependent Bernouilli equation,

$$\frac{\partial V}{\partial t}+\nabla\left(\frac{1}{2}V^2+\frac{p}{\rho_L}\right)=0, \tag{13}$$

applied in the axial direction gives

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\left(\frac{p}{\rho_L}\right)=0. \tag{14}$$

With p/ρ_L given as in eqn.(12), eqn.(14) becomes

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{c^2}{(r_{ac}-\eta)^3}\frac{\partial \eta}{\partial x}=0. \tag{15}$$

The middle term, of eqn.(15), may be omitted as it is second order in the small parameter ε so that it reduces to

$$\frac{\partial u}{\partial t}+\frac{c^2}{(r_{ac}-\eta)^3}\frac{\partial \eta}{\partial x}=0. \tag{16}$$

Now, eqn.(10) may be differentiated w.r.t. t and eqn.(16) differentiated w.r.t. x. With any further nonlinear terms, in the small parameter ε, removed (i.e. the products $\delta\eta/\delta t\ \delta u/\delta x$ and $(\delta\eta/\delta t)^2$) these reduce to

$$\frac{\partial^2\eta}{\partial t^2}+\frac{1}{2}\left[\frac{r_w^2-(r_{ac}-\eta)^2}{r_{ac}-\eta}\right]\frac{\partial^2 u}{\partial x\partial t}=0 \tag{17}$$

and

$$\frac{\partial^2 u}{\partial t\partial x}+\frac{c^2}{(r_{ac}-\eta)^3}\frac{\partial^2\eta}{\partial x^2}=0. \tag{18}$$

By multiplying eqn.(18) through by the coefficient of $\partial^2 u/\partial x\partial t$ from eqn.(17), in square brackets, and then subtracting the result from eqn.(17), so as to remove the $\partial^2 u/\partial x\partial t$ terms, then one obtains

$$\frac{\partial^2\eta}{\partial t^2}-\left[\frac{c^2\left\{r_w^2-(r_{ac}-\eta)^2\right\}}{2(r_{ac}-\eta)^4}\right]\frac{\partial^2\eta}{\partial x^2}=0. \tag{19}$$

As a coefficient, the wave amplitude η is small in comparison to both the air core radius r_{ac} and the nozzle wall radius r_w then the coefficient of $\partial^2\eta/\partial x^2$ may be approximated to give

$$\frac{\partial^2\eta}{\partial t^2}-\left[\frac{c^2\left\{r_w^2-r_{ac}^2\right\}}{2r_{ac}^4}\right]\frac{\partial^2\eta}{\partial x^2}=0. \tag{20}$$

or

$$\frac{\partial^2\eta}{\partial t^2}-\chi^2\frac{\partial^2\eta}{\partial x^2}=0 \tag{21}$$

This is a wave equation with the wave phase velocity given by

$$\chi=\left(\frac{c^2\left(r_w^2-r_{ac}^2\right)}{2r_{ac}^4}\right)^{1/2}. \tag{22}$$

3. Discussion

As observed in [6] this wave phase velocity, eqn.(22) is a similar expression to the critical velocity occurring within the outlet of the swirl atomizer with regard to the principle of maximum flow. There is scope for more work in this type of analysis. One may, for example, seek solutions to eqn.(21) which will provide a knowledge of the wave length, frequency and, amplitude and whether the wave is standing or progressive, and if progressive then in which direction it travels.

4. References

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5. Appendix: The Wave Equation and Phase Velocity for Gravity Waves on a Plane Surface

This analysis is conducted in the cartesian coordinate system which has components x, y, and z with the fluid velocity given by V = u, v, w. The velocity is written as an asymptotic power series in some small parameter ε, which is representative of the maximum wave slope,[5]:

$$\text{i.e.}\quad V = \varepsilon V_0 + \varepsilon^2 V_1 + \varepsilon^3 V_2 + \ldots\ldots \tag{23}$$

In this way the analysis can be thought of as dealing with the first term of the series only. The use of the asymptotic power series then makes nonlinear terms of second order in ε, i.e. of order ε^2, so that they are considered to be negligible in comparison to linear terms. The coordinate system is positioned such that the x and y axes are in the horizontal plane and the positive z axis is pointing vertically upwards, with z = 0 at the mean-free surface of the fluid.

The boundary conditions at the top and bottom surfaces are given by S(x,y,z,t)=0. At the free surface, or water-air interface, the wave height will be at $z = \eta(x, y, t)$ so that $S = \eta - z = 0$. As with the fluid velocity V, and for the same reason, the wave height η is considered to be of an asymptotic power series in the small parameter ε, so that

$$\eta = \varepsilon\eta_0 + \varepsilon^2\eta_1 + \varepsilon^3\eta_2 + \ldots\ldots \tag{24}$$

At the fixed surface or seabed, for ocean waves, z = -h so that S = z + h = 0. The analysis by [1] begins with a general form of h = h(x,y), i.e. the depth of the seabed will differ from place to place. Later [1] makes the simplifying assumption, in developing the wave that the bottom is uniform and that h is in fact an absolute constant. The analysis simplifies considerably if one makes this simplifying assumption from the outset, and so this is done here.

In developing the wave equation, for long waves in shallow water, the following three equations are employed, the total (particle or convective) time derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\underline{V}.\nabla) = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \tag{25}$$

the gradient of the time dependent version of the Bernoulli equation

$$\nabla\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}V^2+\frac{p}{\rho_L}+gz\right)=\frac{\partial V}{\partial t}+\nabla\left(\frac{1}{2}V^2+\frac{p}{\rho_L}+gz\right)=0 \qquad (26)$$

(where φ is the velocity potential) and the equation of continuity, for an incompressible liquid,

$$\nabla.\underline{V}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \qquad (27)$$

As the top and bottom surface boundary conditions are S = 0 then the total derivative of the parameter S, using eqn.(25), gives DS/Dt = 0 thus on S = η - z = 0, or z = η,

$$\frac{DS}{Dt}=\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}+v\frac{\partial\eta}{\partial y}-w=0. \qquad (28)$$

Equation (28) contains two nonlinear terms which are second order in the small parameter ε and so will be negligible in comparison to the remaining terms. As the value of w, in eqn.(28), occurs on the surface, z = η, then it will be denoted as such by the use of a subscript. Hence eqn.(28) may be rewritten as

$$\frac{\partial\eta}{\partial t}=w_{(z=\eta)}. \qquad (29)$$

The continuity equation, (27), may be rewritten as an integral to provide

$$w_{(z=\eta)}-w_{(z=-h)}=-\int_{z=-h}^{z=\eta}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz. \qquad (30)$$

As u and v are independent of z and $w_{(z=-h)}=0$, there is no motion normal to the seabed at the seabed, then eqn.(30) produces

$$w_{(z=\eta)}=-(\eta+h)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right). \qquad (31)$$

As a coefficient η is small in comparison to h, in addition it would produce second order terms in the small parameter ε when multiplied by the velocity gradients, the coefficient η + h may therefore be reduced to h. The resultant expression, on the RHS of eqn.(31), may then replace the RHS of eqn.(29) to give

$$\frac{\partial\eta}{\partial t}+h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0. \qquad (32)$$

This will be left for the moment and attention will be turned to the pressure.

For the water, in a quiescent state, the pressure at a point z below the surface will be given by p = $\rho_L gz$, where ρ_L is the liquid density. If however the surface of the water has a wave of height η then the pressure at a point z + η below this free surface will have a pressure given by

$$p=\rho_L g(z+\eta). \qquad (33)$$

The Bernoulli equation, (26), can be applied to the wave problem with the pressure given as in eqn.(33). As eqn.(33) already contains the gravity term g then the elevation head term in eqn.(26) is taken up by this parameter. Hence the u and v terms of eqn.(26) become

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial\eta}{\partial x}=0 \qquad (34)$$

and

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+g\frac{\partial\eta}{\partial y}=0. \qquad (35)$$

The nonlinear middle terms of both of eqns. (34) and (35) are second order in the small parameter ε and hence negligible so these equations may be rewritten as

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0 \qquad (36)$$

and

$$\frac{\partial v}{\partial t} + g\frac{\partial \eta}{\partial y} = 0. \qquad (37)$$

Next eqn.(32) may be differentiated w.r.t. t, eqn.(36) differentiated w.r.t. x and eqn.(37) differentiated w.r.t. y. The results of these actions may be combined to form

$$\frac{\partial^2 \eta}{\partial t^2} - gh\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) = 0. \qquad (38)$$

This is a 'Wave Equation', it describes how the wave amplitude η behaves both in terms of time t and position in the x, y plane on the water surface. The phase velocity of the wave χ will be given by

$$\chi = \sqrt{(gh)}\ . \qquad (39)$$