Dimensional Analysis and the Buckingham-Π Theorem Applied to the Inviscid Swirl Atomizer Governing Equations

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Abstract

The classical governing equation which describes the internal flow of a pressure swirl atomizer is considered in regard to its non-dimensional form. The non-dimensional parameters (or 'dimensionless groups') used were:- discharge coefficient, dimensionless air core and the atomizer constant. Using the Buckingham-Π theorem, it can be shown that there ought to be a fourth dimensionless group. It is postulated that a two-governing equation model may provide a superior formulation for swirl atomizer inviscid flow and a serious limitation of the classical inviscid theory is indicated.

Introduction

One of the most important parameters of swirl atomizer internal flow is the size of the air core in the outlet orifice, from which may be determined the operating liquid film thickness. From the film thickness, liquid velocities and liquid and ambient gas properties the spray droplet structure may be ascertained. Early work on determining the size of the air core used inviscid methods based on (a) the Bernoulli equation, (b) continuity, and (c) the conservation of angular momentum. Giffen and Muraszew [1] and Abramovich [2] independently formulated a governing equation for the inviscid flow within a swirl atomizer. In full, dimensional, form this is, for [1] and [2] respectively:

$$\frac{Q^2}{\pi^2\left(r_o^2 - r_{oac}^2\right)^2} + \frac{Q^2\, r_s^2}{i^2\, \pi^2\, r_i^4\, r_{oac}^2} = \frac{2\Delta p}{\rho_L} \quad \text{or}$$
$$\frac{Q^2}{\pi^2\left(r_o^2 - r_{oac}^2\right)^2} + \frac{Q^2\, \left(r_s - r_i\right)^2}{i^2\, \pi^2\, r_i^4\, r_{oac}^2} = \frac{2\Delta p}{\rho_L}. \tag{1}$$

Here, Q is the volumetric flow rate; r_o, r_s, r_i, r_{oac} are the radii of, respectively, the outlet, the swirl chamber, the inlets (for round inlets) and the air core in the outlet; i is the number of inlets; ρ_L is the liquid density and Δp is the supply pressure. In dimensionless form the equation is used to determine the discharge coefficient, C_D, and the dimensionless outlet air-core radius, R_o, for a given atomizer constant, K (for [1] or K' for [2]). These dimensionless groups are defined as

$$C_D = \frac{Q}{\pi r_o^2 \sqrt{\dfrac{2\Delta p}{\rho_L}}}, \quad R_o = \frac{r_{oac}}{r_o},$$
$$K = \frac{i\pi r_i^2}{\pi r_o r_s} = \frac{A_i}{\pi r_o r_s} \quad \text{and} \quad K' = \frac{A_i}{\pi r_o \left(r_s - r_i\right)}. \tag{2}$$

These are three dimensionless groups. It was mentioned in an earlier work, Yule and Chinn [3], that if a formal methodology for determining the dimensionless groups, from Eq.(1), is used, then an additional dimensionless group is obtained, N ($N = r_o/r_s$). In the present article this methodology is developed.

Specific Objectives

One-Conservation Equation Model

The dimensional analysis is based on the Buckingham-Π theorem [4] which must be applied to a physical process, i.e. an equation or a system of equations, which satisfies the principle of dimensional homogeneity, that is, all terms in these equations are to be of the same powers of *basic* dimensions: mass, length, time and temperature (MLTΘ). For example, a pressure must be equal to the sum of a number of other pressure terms and a velocity which is divided by a viscosity must be equal to the sum of a number of other terms which have the dimensions of a velocity divided by a viscosity etc. Clearly any other formulation must be incorrect. The theorem states that if the physical process involves an integer number n *primary* physical quantities, for example pressure, length, velocity, viscosity etc, which are either *basic* dimensions themselves or else are made up of the *basic* dimensions, then it can be reduced to a relation between only an integer number k dimensionless groups or Π's. The reduction j = n - k equals the maximum number of *primary* physical quantities which do not form a Π -group among themselves, and is always less than or equal to the number of *basic* dimensions (MLTΘ) involved in the physical process.

Having found the reduction j then one must select j of the primary physical quantities as *scaling* quantities which cannot form a dimensionless Π group. Each of the k dimensionless Π-groups will then be a power product of these j primary scaling quantities together with one of the remaining primary quantities, which is itself raised to a power which is

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optionally chosen to be the most convenient. Using Huntley's addition [5], for the various primary radii, Table 1 gives examples of *basic* dimensions and *primary* quantities for Eq.(1):

Table 1	1	2	3	4	5	6	7
Primary	Δp	ρ_L	Q	r_o	r_s	r_i	r_{oac}
Basic	$ML^{-1}T^{-2}$	ML^{-3}	L^3T^{-1}	L	L	L	L

The primary quantities involve the three basic dimensions of mass, length and time (MLT) and hence $j \le 3$ and it may be seen that it is possible to select three of the primary quantities which cannot form a dimensionless group (or Π) no matter what powers they are raised to. For instance, Δp, ρ_L and r_o will not form a dimensionless group, as only Δp contains T, thus $j = 3$. Therefore, from $j = n - k$, with $j = 3$ and $n = 7$ then $k = 4$, there are to be 4 formal dimensionless groups, or Π's. The choice of the scaling dimensions is arbitrary, providing that they cannot form a Π-group, and one ought to look for those which will form known dimensionless groups such as the discharge coefficient C_D and the atomizer constant K. As Δp, ρ_L and r_o are used to form C_D then these are to be the scaling dimensions used to form the four Π-groups as follows:

$$\begin{aligned}
\Pi_1 &= \Delta p^a \rho_L^b r_o^c Q^{-2} = (ML^{-1}T^{-2})^a (ML^{-3})^b (L)^c (L^3T^{-1})^{-2} = M^0L^0T^0 \\
\Pi_2 &= \Delta p^a \rho_L^b r_o^c r_s^1 = (ML^{-1}T^{-2})^a (ML^{-3})^b (L)^c (L)^{-2} = M^0L^0T^0 \\
\Pi_3 &= \Delta p^a \rho_L^b r_o^c r_i^1 = (ML^{-1}T^{-2})^a (ML^{-3})^b (L)^c (L^2)^{-4} = M^0L^0T^0 \\
\Pi_4 &= \Delta p^a \rho_L^b r_o^c r_{oac}^1 = (ML^{-1}T^{-2})^a (ML^{-3})^b (L)^c (L)^2 = M^0L^0T^0
\end{aligned} \tag{3}$$

These give simultaneous equations in a, b and c for each Π, which give the following results

$$\Pi_1 = \frac{r_o^4 \Delta P}{Q^2 \rho_L},\ \Pi_2 = \frac{r_s}{r_o},\ \Pi_3 = \frac{r_i}{r_o} \text{ and } \quad \Pi_4 = \frac{r_{oac}}{r_o} \tag{4}$$

Π_1 and Π_4 are the variables and Π_2 and Π_3 are based on the dimensions of the atomizer.

Equation (1) may be made dimensionless by multiplying it through by $\pi^2 r_o^4/Q^2$ to give

$$\frac{1}{\left(1 - \frac{r_{oac}^2}{r_o^2}\right)^2} + \frac{r_o^4 r_s^2}{i^2 r_i^4 r_{oac}^2} = \frac{2\Delta p \pi^2 r_o^4}{\rho_L Q^2} \tag{5}$$

which may be written in terms of the Π-groups as

$$\frac{1}{\left(1 - \Pi_4^2\right)^2} + \frac{\Pi_2^2}{i^2 \Pi_3^4 \Pi_4^2} = 2\pi^2 \Pi_1. \tag{6}$$

By following the methodology of the Buckingham-pi theorem through to the letter four formal dimensionless Π-groups were obtained for the equation of Giffen and Muraszew; however an inspection of Eq.(6) reveals that only the groups Π_1 and Π_4 occur independently, and that the ratio Π_2/Π_3 occurring in the second term on the LHS might equally as well have been written as a single group. This was in fact done, less formally, by Giffen and Muraszew [1]. With reference to Eq.(2) for C_D and K, it may be seen that one may write

$$\frac{\Pi_2^2}{i^2 \Pi_3^4} = \frac{1}{K^2} \quad \text{and} \quad 2\pi^2 \Pi_1 = \frac{1}{C_D^2} \tag{7}$$

and with Π_4 replaced by Giffen and Muraszew's own parameter X ($X = \Pi_4^2$) then Eq.(6) may be written as

$$\frac{1}{(1 - X)^2} + \frac{1}{K^2 X} = \frac{1}{C_D^2} \tag{8}$$

which is the dimensionless equation governing the inviscid flow in the outlet orifice of the swirl atomizer of Giffen and Muraszew [1]. This equation will be referred to again presently.

<u>Two-Conservation Equation Model</u>

The axial velocity at the rear wall of the atomizer is zero. Due to conservation of angular momentum, according to inviscid theory, the swirl velocity is constant, for any given radius, throughout the length of the atomizer. Taylor [6] remarked upon this and noted that the air-core diameter at the rear wall ought to be different to that in the outlet orifice. By extension of this notion, as the swirl chamber is larger than the outlet orifice, the axial velocity will also differ there from that in the outlet. The air core in the swirl chamber, r_{sac}, will therefore be different from that in the outlet. Using Abramovich's [2] form of the governing equation, the analogical form of the, similar, governing equation for the swirl chamber will be:

$$\frac{Q^2}{\pi^2\left(r_s^2 - r_{sac}^2\right)^2} + \frac{Q^2\left(r_s - r_i\right)^2}{i^2\pi^2\, r_i^4\, r_{sac}^2} = \frac{2\Delta p}{\rho_L}. \quad (9)$$

This introduces an eighth primary physical quantity, r_{sac}. There are still only three primary dimensions (M, L and T), as indicated in Table2. Therefore, according to the Buckingham-Π theory, there ought to be five Π-groups. The additional one being $\Pi_5 = r_{sac}/r_s$:

$$\Pi_1 = \frac{r_o^4 \Delta P}{Q^2 \rho_L},\ \Pi_2 = \frac{r_s}{r_o},\ \Pi_3 = \frac{r_i}{r_o},\ \Pi_4 = \frac{r_{oac}}{r_o} \text{ and } \Pi_5 = \frac{r_{sac}}{r_s}. \quad (10)$$

Table 2	1	2	3	4	5	6	7	8
Primary	Δp	ρ_L	Q	r_o	r_s	r_i	r_{oac}	r_{sac}
Basic	$ML^{-1}T^{-2}$	ML^{-3}	L^3T^{-1}	L	L	L	L	L

A system of two governing equations may therefore be conceived; one in the outlet and one in the swirl chamber. Abramovich's form of the governing equation in the outlet, the second form indicated in Eq.(1), along with the second form of the atomizer constant K, from Eq.(2) will be used here. Eq.(9) has a similar form, i.e. with $r_s - r_i$ in the second term on the left rather than just r_s.

By multiplying both Eq.(1) and Eq.(9) through by $\pi^2 r_o^4/Q^2$ one obtains

$$\frac{1}{\left(1 - \frac{r_{oac}^2}{r_o^2}\right)^2} + \frac{r_o^4\left(\frac{r_s}{r_o} - \frac{r_i}{r_o}\right)^2}{i^2 r_i^4 \frac{r_{oac}^2}{r_o^2}} = \frac{2\pi^2 r_o^4 \Delta p}{Q^2 \rho_L} \text{ and } \quad (11)$$

$$\frac{1}{\left(\frac{r_s^2}{r_o^2} - \frac{r_s^2}{r_o^2}\frac{r_{sac}^2}{r_s^2}\right)^2} + \frac{r_o^4\left(\frac{r_s}{r_o} - \frac{r_i}{r_o}\right)^2}{i^2 r_i^4 \frac{r_s^2}{r_o^2}\frac{r_{sac}^2}{r_s^2}} = \frac{2\pi^2 r_o^4 \Delta p}{Q^2 \rho_L} \quad (12)$$

which may be written in terms of the five Π-groups as

$$\frac{1}{\left(1 - \Pi_4^2\right)^2} + \frac{\left(\Pi_2 - \Pi_3\right)^2}{i^2\, \Pi_3^4\, \Pi_4^2} = 2\pi^2\, \Pi_1 \text{ and } \quad (13)$$

$$\frac{1}{\Pi_2^4\left(1 - \Pi_5^2\right)^2} + \frac{\left(\Pi_2 - \Pi_3\right)^2}{i^2\, \Pi_3^4\, \Pi_2^2\, \Pi_5^2} = 2\pi^2\, \Pi_1. \quad (14)$$

These two equations may be written in terms of the more conventional dimensionless groups C_D and K'

$$\frac{\left(\Pi_2 - \Pi_3\right)^2}{i^2\, \Pi_3^4} = \frac{1}{K'^2} \text{ and } 2\pi^2\, \Pi_1 = \frac{1}{C_D^2} \quad (15)$$

together with the new dimensionless group N,

$$N = \frac{r_o}{r_s} = \frac{1}{\Pi_2}. \quad (16)$$

and the dimensionless air-core radii groups, which are to be defined as

$$R_o = \frac{r_{oac}}{r_o} = \Pi_4 \text{ and } R_s = \frac{r_{sac}}{r_s} = \Pi_5 \quad (17)$$

to give

$$\frac{1}{\left(1 - R_o^2\right)^2} + \frac{1}{K'\, R_o^2} = \frac{1}{C_D^2} \text{ and } \quad (18)$$

$$\frac{N^4}{\left(1 - R_s^2\right)^2} + \frac{N^2}{K'\, R_s^2} = \frac{1}{C_D^2} \quad (19)$$

from Eq.(13) and Eq.(14), respectively

Discussion

The four formal dimensionless Π-groups which were used in making the equation of Giffen and Muraszew dimensionless were reduced to three when the conventional dimensionless groups, C_D, K and R_o, from Eq.(2), were used, whilst, in using a system of two equations to describe the inviscid flow within the swirl atomizer, the number of formal dimensionless groups, Π_1 to Π_5, is retained in the five 'conventional' dimensionless groups, C_D, K', N, R_o and R_s. The omission of the parameter N, by Giffen and Muraszew, occurs because their analysis is limited to the situation in the outlet of the atomizer, and as a result only one governing equation for the flow within the atomizer is assumed, Eq.(8). It is only when both the flow in the outlet and in the swirl chamber is considered, that the existence of the parameter N is made known.

To illustrate the importance of the Π-group N, Fig.(1) shows the internal geometry of four swirl atomizers drawn to the same scale. They all share the same outlet orifice radius size, r_o. The dimensions of these atomizers, in some arbitrary units, are given in Table 3. Atomizers (a) and (b) share the same value of K but have different values for r_i and r_s. Clearly with K and r_o the same then the ratio $i\, r_i^2/r_s$ must be the same for both atomizers. Both r_i and r_s are increased for the atomizer of case (b). The same concept is used for atomizers (c) and (d), which also

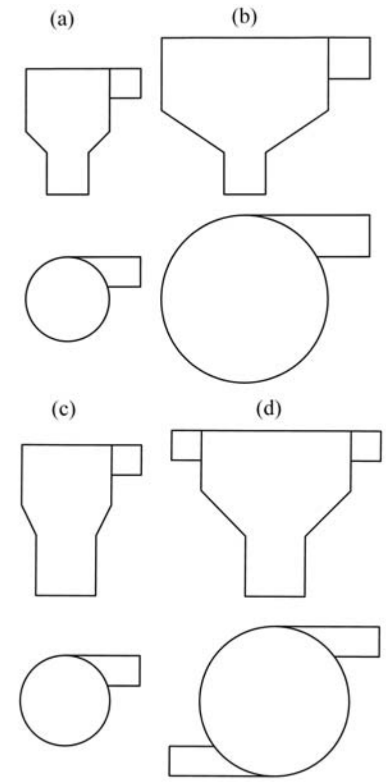

Figure 1. Atomizers (a) and (b) share the same K value, but different N. Atomizers (c) and (d) share the same K' value but different N.

share the same r_o value but this time share the same K' value. The ration $i\,r_i^2/\left(r_s - r_i\right)$ is the same for both (c) and (d) and again, both the inlet area cross-section and the swirl chamber size are increased in (d).

The classical inviscid treatment would have that atomizers (a) and (b) share the same value of the discharge coefficient C_D and dimensionless air core radius in the outlet, R_o. The same goes for atomizers (c) and (d), which share the same K' value. Kutty et al. [7] and Dombrowski and Hasson [8] have shown experimentally that both C_D and R_o will indeed be different for atomizers having similar K values but different values of the parameter N. Experimentally, it was shown that for any given K or K' then R_o increases as N increases. For any given K or K' then C_D deceases as N increases. Kutty et al. [7] states that air core diameter increases with a decrease in inlet size. Clearly this would be due to a restriction in the flow which would lead to a thinner film thickness in the outlet. Kutty et al. [7] also says that the air core diameter increases with an increase in swirl chamber diameter, up to a particular value and afterwards remains substantially constant. The data from [7] is re-plotted in the two charts of Fig.(2), for R_o vs. K', and Fig.(3), and C_D vs. K'. Both clearly indicate the dependence of these parameters on the dimensionless group N.

Table 3	(a)	(b)	(c)	(d)
r_o	10	10	10	10
i	1	1	1	2
r_i	5	$5\sqrt{2}$	5	5
r_s	20	40	15	25
K	0.125	0.125	-	-
K'	-	-	0.25	0.25
N	0.5	0.25	0.667	0.4

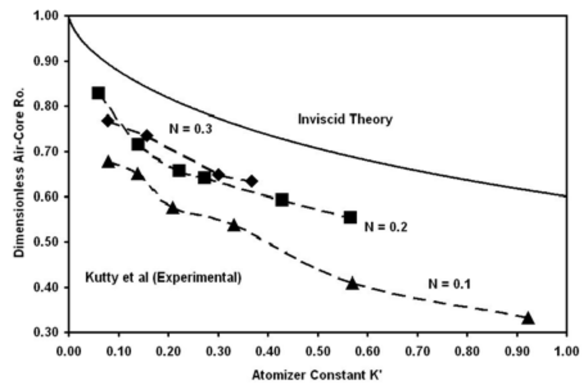

Figure 2. The variation of the dimensionless air core radius, R_o with the atomizer constant K' and Π-group N.

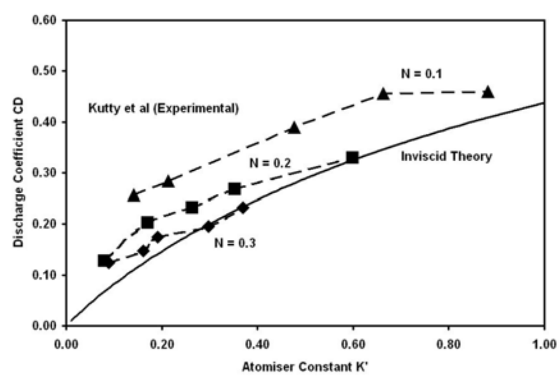

Figure 3. The variation of the discharge coefficient C_D with the atomizer constant K' and Π-group N.

Conclusion

While the classical inviscid theory appears to be mathematically sound, on the assumptions on which it is based, the assumptions themselves are more simplistic than they need to be and the problem of formulating an inviscid analysis will benefit from the inclusion of the additional independent parameter N.

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