SWIRL ATOMIZER FLOW: CLASSICAL INVISCID THEORY REVISITED

A. J. Yule, J. J. Chinn
University of Manchester Institute of Science and Technology, UMIST, Manchester, England.

ABSTRACT
The much quoted, classical theory for swirl atomizer performance requires the discharge coefficient and spray angle to be dependent only upon a dimensionless group \( K = A_i / \{(d_i - d_o)d_o\} \). Observed deviations from the ideal theory have been assumed, in the past, to be caused by "real effects", such as viscosity and surface tension. Additional corrections have been proffered, in particular, using the ratio \( d_i/d_o \). The authors note that dimensional analysis shows that inviscid atomizer performance cannot depend only upon one group, \( K \), and that an additional group should be specified. Classical inviscid analysis results in a dependency only upon \( K \) due to simplifications used to obtain relationships for the air-core diameter. Hitherto inviscid analysis has discharge coefficient \( C_D \) based on a sole equation, dependent upon at least two independent variables. The dependence is then simplified to \( K \) as single independent group under the assumption that the air-core radius will adjust for maximum discharge and a differentiation, of the expression, was then undertaken. These simplifications were used because of the failure to apply a further conservation equation to the flow: The conservation of axial momentum. This conservation equation has been applied by the authors and a new inviscid solution has been derived showing atomizer performance to depend upon both \( K \) and the additional group \( d_i/d_o \). The results of this analysis are compared with "old" classical theory and experimental results. The theory provides a sounder basis for the performance analysis and design of pressure swirl atomizers and forms a foundation for making enhancements, such as viscous boundary effects.

INTRODUCTION
Existing Theory, Based on Bernoulli’s Equation Applied in the Outlet. What will be referred to as the "classical inviscid theory", for the flow in swirl atomizers, appears to have been developed at least 50 years ago; Giffen and Muraszew[1], Taylor[2]. This theory combines the conservation of angular momentum (derivable from the \( r \) and \( x \) components of the vorticity equation \( \nabla \times \mathbf{U} = 0 \)), the conservation of mass flow and Bernoulli’s equation, assuming an internal geometry similar to that shown in figure 1. A free vortex is assumed to occur in the swirl chamber and an air-core forms where the pressure is atmospheric. Bernoulli’s equation is:

\[
\frac{u^2 + v^2 + w^2}{2} + \frac{p}{\rho} = \frac{P_i}{\rho}.
\]

The distribution \( w(r) \) is identical throughout the atomizer and

\[
w_r = Q_r \frac{(r_s - r_i)}{A_i}.
\]

Within the nozzle outlet orifice one has \( u_o = Q(\tau_o - \tau_{oac})/A_{i} \) and evaluating Eq. (1) at the inner surface of the air-core, within the outlet, where \( P = 0 \), gives:

\[
\frac{Q^2}{2\pi^2(\tau_o^2 - \tau_{oac}^2)^2} + \frac{Q^2(r_s - r_i)^2}{2A_i^2r_{oac}^2} = \frac{P_i}{\rho}.
\]

As shown in Fig. 1 for an inviscid flow, the axial \( u \) velocity is independent of \( r \), both in the swirl chamber and in the nozzle outlet. This can be shown from the \( \theta \) component of the vorticity equation together with the assumption of negligible radial velocity \( v \) in the outlet.
Given a specified supply pressure $P_i$, Eq. (3) contains two unknowns, $Q$ and $r_{osc}$. Alternatively one may re-write Eq. (3) in terms of $C_D$ and $r_{osc}$ (or $C_D$ and $A_4$ where $A_4$ is the film cross-sectional area). Thus an additional assumption (or equation) is required to obtain a solution. Both Taylor[2] and Giffen and Muraszew[1] made the assumption that, for any given injection pressure $P_i$ the air-core radius in the outlet $r_{osc}$ will always be such that volumetric flow rate $Q$ will be a maximum, $\partial Q/\partial r_{osc} = 0$. In this way it becomes possible to form an equation relating the discharge coefficient $C_D$ directly as a function of air-core radius $r_{osc}$ and indirectly as a function of atomizer constant $K$. Although Taylor[2] justified this approximation by analogy with choking occurring at the minimum cross-section of a nozzle in high speed gas flow, this analogy is not exact.

**Reasons for Seeking a New Theory.** The limitation of the above approach is that it ignores the concept of axial momentum conservation. If axial momentum balance is combined with the Bernoulli equation, which is essentially a specific energy equation, in the form of Eq. (3), then it is argued here that a much more rigorous approach results. It will be found that a consequence of deriving the equation governing axial momentum is that another dimensionless parameter, $N = d_o/d_s$, emerges to complement the atomizer constant $K$. It was recognised by several authors, working experimentally, that to have $C_D$ dependent upon $K$ alone was insufficient and a further dimensionless group was sought: Dombrowski and Hasson[3] used

$$\frac{A_4}{d_s^2 d_o} \left( \frac{d_s}{d_o} \right)^{1/2}$$

as the independent variable against which to plot empirically derived values of $C_D$. Rizk and Lefebvre[4] proposed

$$C_D = 0.35 \left( \frac{A_4}{d_s^2 d_o} \right)^{1/2} \left( \frac{d_s}{d_o} \right)^{1/4}$$

which is also based on empirical data. Lefebvre[5] discusses

THE NEW THEORY

**The Dimensionless Group N** It is not surprising that two dimensionless parameters are required to describe the effects of atomizer geometry. For the inviscid quasi-one-dimensional flow of classical theory (for which the lengths of the swirl chamber and nozzle outlet have no effects) the basic parameters describing the flow rate/geometry interrelationship are $P_i$, $Q$, $\rho$, $A_i$, $d_s$ and $d_o$ (and also the number of inlets, $n$: but this is subsumed in the parameter $A_i = n \pi r_i^2$, in the case of round inlets, in the parameter $K$). With the three dimensions of mass, length and time Buckingham's "PI" theorem requires there to be three dimensionless groups and these may be chosen to be $C_D$, $K$ and $N$. The simplification in the "classical" theory, described in the introduction, results in the loss of the group $N$.

**A Further Mathematical Description of the Flow** Figure 2 indicates that the conservation of axial momentum requires equating the reaction force $F_{net}$ (balancing the exit axial momentum) to the axial force on the nozzle

---

**Figure 1.** Sections of the internal geometry of a typical two-inlet pressure swirl atomizer showing nomenclature and inviscid velocity profiles.
body produced by integrating the wall static pressure distribution at the inner surface of the nozzle. This requires a full description of the internal flow pattern expected for an inviscid flow. As indicated in Fig. 1 the velocity field and air-core radius vary inside the atomizer. The variation of $w$ is known everywhere from Eq. (2). However to obtain $u$, one must know how the air core varies inside the atomizer.

Existing inviscid or potential theory is based on a single Bernoulli equation, applied at the atomizer outlet, which has the discharge coefficient $C_D$ as a function of both the atomizer constant $K$ and the dimensionless air-core in the outlet. Without undertaking the differentiation, which was described earlier, this single equation is unsolvable as it contains too many unknowns ($C_D$ and $r_{onc}$). The present authors dispute the validity of performing this differentiation as there appear to be little grounds for doing so apart from a loose entropy argument. To obtain a solution for the inviscid flow would therefore require the introduction of a further equation. To facilitate this, firstly a second Bernoulli equation is constructed, this time describing the flow within the swirl chamber. The two Bernoulli equations combine to eliminate the discharge coefficient $C_D$ but have, instead, the air-core radii in the outlet and in the swirl chamber ($r_{onc}$ and $r_{swc}$) as unknowns. Attention is then drawn to the momentum balance so that the necessary additional equation is introduced. This equation is formulated by considering the balance of forces at work within the atomizer: There is a pressure acting on the top face of the swirl chamber and on the conical convergence of the atomizer between the swirl chamber and the outlet The resultant of these opposing forces is the axial momentum transport of the fluid as it exits the outlet orifice. This additional equation also has the two air-cores as its unknowns. Having determined the air-core radii in the outlet and in the swirl chamber the remaining air-core radius at the top of the swirl chamber, $r_{onc}$, and the discharge coefficient $C_D$ are readily determined. Also a knowledge of the air core radii allows determination of the axial velocity $u$.

An Additional Bernoulli Equation in the Swirl Chamber. A similar equation to Eq. (3) may be derived in the swirl chamber to determine the radius of the air-core there, $r_{swc}$:

$$\frac{Q^2}{2\pi^2(\frac{z^2}{R_{onc}^2} - \frac{z^2}{R_{swc}^2})^2} + \frac{\rho^2}{2\pi^2(r_{onc}^2 - r_{swc}^2)^2} = \frac{P}{\rho}.$$  

(6)

Here again it is assumed that the radial velocity $v$ is negligible. Giffen and Muraszew non-dimensionalised Eq. (3) by the use of three dimensionless quantities; $C_D$, $K$ and $X = r_{onc}/r_{o}$. However, it is preferred to use $r_o$ as the length scale to non-dimensionalise the air-core in both the outlet and in the swirl chamber. This will necessitate the introduction of the group $N = d_i/d_o$. Thus both Eq. (3) and Eq. (6) may be non-dimensionalized to form

$$\frac{N^2}{K^2} C_D = \frac{R_o^2(N^2 - R_o^2)^2}{(N^2 - R_o)^2 + R_o^2 N^2 K^2}$$  

(7)

and

$$\frac{N^2}{K^2} C_D = \frac{R_o^2(1 - R_o^2)^2}{(1 - R_o^2)^2 + R_o^2 N^2 K^2}.$$  

(8)
Where, for convenience, an alternative atomizer constant is used; \( K_\text{t} = 4K/\pi \). Eqs. (7) and (8) are equal and may be combined to form one equation in the two unknowns; \( R_o = r_{\text{osc}}/r_\text{t} \) and \( R_\text{t} = r_{\text{osc}}/r_\text{s} \).

**The Air-Core at the Top of the Swirl Chamber.** At the "top" face of the atomizer (in Figs. 1 and 2) the axial and radial velocity components are zero so that here Bernoulli’s equation becomes

\[
\frac{w_\text{r}}{2} + \frac{p}{\rho} = \frac{p_\text{l}}{\rho}.
\]  

(9)

At the air-core, where the pressure is atmospheric, Eq. (9) becomes \( w^2/2 = p_\text{l}/\rho \). Thus, using Eq. (2), one has

\[
\frac{Q^2(r_\text{s} - r_\text{t})^2}{2A_\text{f}^2 r_\text{osc}^2} = \frac{p_\text{l}}{\rho}.
\]

(10)

This may be non-dimensionalised to form

\[
R_\text{t}^2 = \frac{N^2}{K_\text{t}^2} C_\rho^2
\]

(11)

where \( R_\text{t} = r_{\text{osc}}/r_\text{s} \), the dimensionless air-core at the top of the swirl chamber. Eq. (11) should be compared with Eqs. (7) and (8).

**The Axial Momentum Conservation Equation, A Simple Case** In order to introduce a further equation consideration is made of the axial momentum balance. As indicated in Fig. 2 the pressure force of the fluid acting on the top face of the atomizer is balanced by a sum of quantities consisting of: the pressure force on the convergence wall of the atomizer, the force due to the pressure variation across the annular liquid film in the outlet orifice and the axial momentum rate of this annulus of fluid \( \rho Qu_\text{o} \). As a demonstration of a simplified case consideration is first made of an atomizer with no conical wall i.e where \( d_\text{s} = d_\text{o} \).

At the top face Eq. (9) is rearranged to make \( P \) the subject and this is then integrated from \( r_{\text{osc}} \) (where \( P = 0 \)) to \( r_\text{t} \) to determine the force \( F_\text{t} \) on this face. This force is made dimensionless by dividing by \( \pi P_\text{s} r_\text{t}^2 \) to form

\[
\frac{F_\text{t}}{\pi P_\text{s} r_\text{t}^2} = \Phi_\text{t} = 1 - R_\text{t}^2 + R_\text{t}^2 \ln R_\text{t}^2.
\]

(12)

Next the force on the annular liquid film in the outlet is obtained by making \( P \) the subject of Eq. (1), with \( v \) negligible, and integrating this from \( r_{\text{osc}} \) to \( r_\text{t} \) so as to determine \( F_\text{o} \). This may also be non-dimensionalized by dividing by \( \pi r_\text{s}^2 F_\text{l} \) to form

\[
\frac{F_\text{o}}{\pi r_\text{s}^2 F_\text{l}} = \Phi_\text{o} = N^2 - R_\text{o}^2 - 2R_\text{t}^2 \ln \left( \frac{N}{R_\text{o}} \right) - \frac{R_\text{t}^2 N^2 K_\text{t}^2}{(N^2 - R_\text{o}^2)}
\]

(13)

where \( R_\text{o} = r_{\text{osc}}/r_\text{t} \).

Finally the axial momentum transport \( \rho Qu_\text{o} \) is non-dimensionalized to form

\[
\frac{\rho Q u_\text{o}}{\pi r_\text{s}^2 F_\text{l}} = \Phi_\text{m} = \frac{2R_\text{t}^2 N^2 K_\text{t}^2}{(N^2 - R_\text{o}^2)}
\]

(14)

The axial momentum balance is thus, for this simplified case with \( d_\text{o} = d_\text{s} \) (ie \( N = 1 \)):

\[
\Phi_\text{o} + \Phi_\text{m} - \Phi_\text{t} = R_\text{t}^2 - R_\text{o}^2 + R_\text{t}^2 \ln \left( \frac{R_\text{o}^2}{R_\text{t}^2} \right) + \frac{R_\text{t}^2 K_\text{t}^2}{(1 - R_\text{o}^2)} = 0.
\]

(15)

By using Eqs. (7) and (11) in Eq. (15), an equation in \( R_\text{o} \) only may be formed. To determine \( R_\text{o} \) one may employ a numerical scheme such as the Newton-Raphson method or a bisection algorithm. Thus, on determining \( R_\text{o} \), one may also determine \( R_\text{t} \) and \( C_\text{D} \).

**The Case With a Convergence** The realistic case of an atomizer having \( d_\text{o} > d_\text{s} \) will require an expression for the additional downward force, in Fig. 2, on the convergence of the atomizer, \( F_\text{c} \). This is the force, acting on the cone, resolved into the axial direction and it can be shown that the magnitude of this force will be the
integral of Eq. (1), with P as subject, with respect to the radius of the cone, \( r_c \), from \( r_a \) to \( r_e \), ie from the outlet radius to the radius of the swirl chamber:

\[
F_c = 2\pi \int_{r_a}^{r_e} r_c \, dp \, dr_c = 2\pi \int_{r_a}^{r_e} r_c \left\{ P_c - \frac{p}{2}(u^2 + v^2 + w^2) \right\} \, dr_c. \tag{16}
\]

The variation of the radius of the air core in the conical section of the atomizer, from the outlet to the swirl chamber, is modelled linearly. For a one-dimensional inviscid flow the angle of the convergence is irrelevant and for convenience one may take the case of a very long convergence such that \( u \gg v \) and \( w \gg v \) so that \( v \) may be ignored in Eq. (16). Thus Eq. (16) becomes

\[
F_c = 2\pi \int_{r_a}^{r_e} \left[ P_i - \frac{p \rho^2}{2\pi^2 r_c^3} \left( \frac{r_c^2}{r_e^2} \left( I_{\text{vac}} - I_{\text{vac}} \right) \left( I_c - I_0 \right) + I_{\text{vac}} \right) \right] \, dr_c. \tag{17}
\]

By introducing a dimensionless variable \( R_c = r_c / r_a \), Eq. (17) may be rewritten

\[
\Phi_c = \frac{F_c}{\pi r_a^2 P_i} = 2 \int_0^1 \left[ R_c - (N K) \right] \frac{R_c}{1 - N} - \frac{R_o}{N^2 - R_o^2} \, dR_c. \tag{18}
\]

It is possible to solve the equation analytically to give:

\[
\Phi_c = (1 - N^2) + (R_o N K) \left( \frac{1 - N}{N R_a - R_o} \right) \left( \frac{R_a}{1 - R_a^2} - \frac{R_o}{N^2 - R_o^2} \right) + \frac{(R_o - R_a)(1 - N)}{2(N R_a - R_o)^2} \ln \left( \frac{(N - R_o)(1 + R_o)}{(N + R_o)(1 - R_o)} \right) + 2R_e^2 \ln N. \tag{19}
\]

The momentum balance for the atomizer with a conical convergence will thus be

\[
\Phi_o + \Phi_m + \Phi_c - \Phi_e = 0. \tag{20}
\]

With the other dimensionless forces, as defined previously, this expression may be written in full:

\[
(R_o N K)^2 \left( \frac{1}{(N^2 - R_a^2)} - \frac{1 - N}{N R_a - R_o} \right) \left( \frac{R_a}{1 - R_a^2} - \frac{R_o}{N^2 - R_o^2} \right) - \frac{(1 - N)}{2(N R_a - R_o)^2} (R_o - R_a) \ln \left( \frac{(N - R_o)(1 + R_o)}{(N + R_o)(1 - R_o)} \right) + 2R_e^2 \ln \left( \frac{R_e}{R_o} \right) - R_a^2 + R_e^2 = 0. \tag{21}
\]

With \( R_i \) replaced by Eqs. (7) (or Eq. (8)) and (11), Eq. (21) has only the two unknowns \( R_o \) and \( R_e \). So that Eq. (21) together with,

\[
\frac{R_o^2(N^2 - R_a^2)}{(N^2 - R_a^2)^2 + R_o^2 N^2 K_1^2} - \frac{R_e^2(1 - R_a^2)^2}{(1 - R_a^2)^2 + R_e^2 N^2 K_1^2} = 0 \tag{22}
\]

ie Eqs. (7) and (8) combined, form a system of two equations in two unknowns.

**SOLUTIONS AND COMPARISON WITH EXPERIMENT**

The system of two equations, Eqs. (21) and (22), in two unknowns may be solved either numerically, eg by use of a two-dimensional Newton-Raphson scheme, or Graphically, by the trial and error testing of one parameter to ensure that the graphs of both functions display the same root in the other parameter. Having determined \( R_o \),
and \( R_s \), then \( C_D \) may also be found from using Eq. (7) or (8) and \( R_s \) may then be determined by Eq. (11). In order to test the validity of the new theory to establish its appropriateness as a design tool a comparison is made with firstly, the existing theory of Taylor and, secondly with the experimental work of Dombrowski and Hasun[3]. Figure 3 gives examples of the solutions for \( C_D \) versus \( K \) for different values of \( N \). The "classical" Taylor theory for the discharge coefficient as a function of atomizer constant is also included. Although the trends of the classical theory are reproduced by the new theory in Fig. 3, it can be seen that the theory of Taylor agrees with the new theory only for large values of \( N \), the limitation of the classical theory in taking no account of the parameter \( N \) is clearly demonstrated.

Another parameter of interest is the spray half-angle \( \alpha \). When the fluid exits the atomizer its formation into a conical sheet, prior to disintegrating into a spray, is not an instantaneous phenomenon and consideration need be made of the situation a little downstream of the outlet, where the spray cone is fully developed. Here the radial velocity component \( v \) will be approximately equal to the tangential velocity \( w_0 \) at the mean annular radius, \( (r_o + r_{oa})/2 \), in the outlet. The overall velocity \( U = \sqrt{2P_i/\rho} \) is constant and

\[
\sin \alpha = \frac{w}{U} = \frac{Q (r_s - r_i)}{A_i (r_o + r_{oa})/2} = \frac{2NC_D}{\pi r_o^2 C_D} \frac{2NC_D}{K (N + R_o)}.
\]

(23)

Giffen and Muraszew[1] state the same result with a different methodology for determining the mean tangential velocity.

A thorough experimental analysis of a range of pressure swirl atomizers was carried out on low viscosity fluids by Dombrowski and Hassun[3]. They have produced graphs of discharge coefficient \( C_D \) and spray angle \( \alpha \) as functions of atomizer constant \( K \), for different values of nozzle parameter \( N \). By using this data it has been possible to make a comparison of the present theory with existing empirical data. In making such a comparison it was made apparent that there should be limitations on the range of values for the parameters employed. We have imposed: \( 0.1 \leq N \leq 1, \ 1 \leq n \leq 9, \ 0.1 \leq S \leq 0.25 \) and \( n \leq \pi N/S \) (for both square and round inlets). The imposition of \( n = 9 \) is made by Dombrowski and Hassun[3]. In practice this imposes an upper limit to \( K \) of around 5 although the results shown here are restricted to about 1.1, in line with the experimental work in [3]. A useful formulation of \( K \), for design analysis, is its formulation in terms of other non-dimensional quantities so that

\[
K = \frac{\pi n S^2}{(1 - S) N}
\]

(24)

for square inlets and

\[
K = \frac{\pi n S^2}{4 (1 - S) N}
\]

(25)

for round inlets.

Figure 3. A comparison of the new theory with that of Taylor.

Figure 4. Shows the variation of \( C_D \) with \( K \). Moving
from bottom-left to top-right the thick lines, represent the theory, with $N = 0.3333$ and $N = 0.1111$. Figure 5 shows the measured values for $\alpha$ together with the present theory, as functions of $K$. In general there is some agreement in trends. However one should not expect a particularly good agreement due to the complete absence of "real" fluid affects in the theory.

The main advantage of this inviscid model is the scope for further improvement by including viscous effects, by, for example, the addition of terms in the conservation relationships for axial and angular momentum. This type of modification could not be achieved satisfactorily with the "classical" theory due to the simplifications which have been discussed. Having now recognised the role of the axial momentum balance it is also possible to extend this simplified inviscid approach by using potential flow theory, for example this direction has been explored partially by Dumouchel et al[6] using the stream function-vorticity formulation.

CONCLUSIONS

By using the conservation equations for axial momentum, the inviscid idealised performance of a swirl atomizer may be predicted without resorting to the simplifications used in earlier analysis. Discharge coefficient and spray angle are found to depend on both the atomizer constant $K$ and also (more weakly) on the diameter ratio $N = d_2/d_0$. The modified approach provides the basis for the inclusion of viscous effects and improved prediction methods for swirl atomizer performance.
NOMENCLATURE

A  cross-sectional area
C_d  discharge coefficient
d  diameter
K  atomizer constant  \( A/(d_1 - d_2)d_1 \)
K_n  = \( A/(r_1 - r_0)\pi r_0^2 \) = \( 4K/\pi \)
n  number of inlets
N  \( d_1/d_2 \)
P  pressure (gauge)
Q  volumetric flow rate
r  radial cylindrical coordinate
R  radius non-dimensionalised
eg \( R_0 = r_{oc}/r_0 \)
S  Ratio of diameters \( d/d_1 \)
u  axial velocity
U  total velocity at exit \( \sqrt{2P/\rho} \)
v  radial velocity
x  axial cylindrical coordinate
\( \alpha \)  spray cone angle
\( \rho \)  liquid density
\( \theta \)  angular cylindrical coordinate
w  tangential, azimuthal or swirl velocity

Subscripts

ac  air-core
c  cone (of the atomizer)
f  liquid film in the outlet
i  inlet
m  momentum
o  outlets
s  swirl chamber
t  top (inlet end of swirl chamber)
\( l \)  Denotes the alternative K

REFERENCES

ACKNOWLEDGEMENTS
The authors thank Wormald Ansell Fire-Protection systems Ltd, Stockport, and The Science and Engineering Research Council (S. E. R. C.) for their financial assistance for this research.